OPEN ACCESS
INTERNATIONAL JOURNAL
EIJMRMS
Next Scientists
European International Journal of Multidisciplinary Research and Management Studies
SJIF 5.954 2022

EUROPEAN INTERNATIONAL JOURNAL OF MULTIDISCIPLINARY RESEARCH AND MANAGEMENT STUDIES
ISSN: 2750-8587
DOI: https://doi.org/10.55640/eijmrms-02-10-12
https://eipublication.com/index.php/eijmrms
Volume: 02 Issue: 10 October 2022
Published Date: - 12-10-2022

Page No.-68-75

STABILITY OF THE EQUILIBRIUM STATE FOR THE SCALAR CONSERVATION LAW WITH NONLOCAL CHARACTERISTIC VELOCITIES

Alimova Vasila Bahodirovna

Assistant, National University Of Uzbekistan

ABSTRACT: - In this paper, we study input-state stability (ISS) of equilibrium for a scalar conservation law with nonlocal velocity and measurement error occurring in a high-volume reproducible system . Using the corresponding Lyapunov function, we derive sufficient and necessary conditions on ISS. We propose a numerical discretization of the scalar conservation law with nonlocal velocity and measurement error. For the proposed numerical approximation, the appropriate discrete Lyapunov function is analyzed to provide the ISS of the discrete equilibrium. For the proposed numerical approximation, the appropriate discrete Lyapunov function is analyzed to provide the ISS of the discrete equilibrium. Finally, we show computational results to confirm the theoretical conclusions.

KEYWORDS: Stabilization of feedback; input-state stagnation; nonlocal velocity.

INTRODUCTION

Items that go through many production steps . This type of production system has fluid-like properties and has been successfully modeled by continuum models [1-5]. In these models, the output and production rate at different production stages are the quantities of interest.

Involving a high-level processing system where products visit the machines several times , such as semiconductor device manufacturing, [3] introduced a Lighthill-Whitham-inspired continuum model - motion model [6]. The dynamics of this model is given mathematically by a hyperbolic partial differential equation of the form:

$$\partial_t\rho(t,x)+\lambda\big(W(t)\big)\partial_x\rho(t,x)=0,\quad t\in[0;\ +\infty),\ x\in[0;1],\qquad (1)$$

here - _$\rho(t,x)$ $t$product density describing x —the total mass in time , $W(t)$production stage,

$$W(t)=\int_0^1\rho(t,x)dx,\qquad t\in(0;+\infty)(2)$$

Unlike classical transport flow models, the differential equation depends on the nonlocal quantity (2).$\lambda\big(W(t)\big)$ function is the characteristic speed. In production systems, it is natural to assume that the velocity function is positive and that the velocity decreases as the total mass increases. In the production system, the initial density of products at the production stage is taken as initial data :x

$$\rho(0,x)=\rho_0(x),\quad x\epsilon[0;1](3)$$

and current is used to control the system or to stabilize the system at equilibrium. Since the velocity is positive, we only $x=0$require the boundary conditions at , i.e., the flow is defined as:

$$\rho(t,0)\lambda\big(W(t)\big)=U(t),\qquad t\in[0;+\infty)(4)$$

λ,ρ_0 and Uunder appropriate assumptions on , the existence and uniqueness of the classical solution of the Cauchy problem for the scalar conservation law of equation (1) with equations (3) and (4) was proved in [7-10].

In this work, our focus is on hyperbolic problems. In connection with the nonlocal velocity (hyperbolic) scalar conservation law, in [10] the authors studied the global feedback stability of the closed loop system in equation (1) under the feedback law.

$$U(t)-\rho^*\lambda(\rho^*)=k\left(\rho(t,1)\lambda\big(W(t)\big)-\rho^*\lambda(\rho^*)\right),\quad t\epsilon(0,+\infty)(5)$$

where is the feedback $k\in[0;1)$parameter and $\rho^*\in R$is the given equilibrium. They generalize the stability results of [7] using the Lyapunov function. In particular, for a given equilibrium $\rho^*=0$and total velocity function , the global stabilization result L^p $(p\geq 1)$for $\lambda\in C^1([0,+\infty);\ [0,+\infty))$the closed-loop system of equations (1), (3) and (5) is L^2generalized to . Then, as a result of global stagnation, a family of velocity functions is derived for the closed-loop system of equations (1), (3) $L^2$and (5) .

$$\lambda(s)=\frac{A}{B+s},\ \ s\in[0,+\infty)\ \ \text{in this}\ \ A>0,B>0(6)$$

the given equilibrium $\rho^*>0$. Using a discrete Lyapunov function, they also established stability results for a discrete scalar conservation law with nonlocal velocity and using a first-order finite volume scheme.

In this paper, we study the ISS for the closed-loop system of equations (1) and (3).

$$U(t) - \rho^*\lambda(\rho^*) = k\left(\rho(t,1) + d(t)\lambda\big(W(t)\big) - \rho^*\lambda(\rho^*)\right), t\epsilon(0,+\infty) \quad (7)$$

where is the bounded $d(t) \in R$fluctuation in the scale. In particular, $\rho^* \geq 0$ we use the ISS-Lyapunov function to check the sufficient and necessary conditions for ISS for equilibrium and the velocity function defined by equation (6) .L^2

2. Stability of the scalar conservation law with nonlocal velocity and measurement error.

Study the ISS of a closed-loop system of scalar conservation laws with nonlocal velocity and measurement error:

$$\begin{cases} \partial_t\rho(t,x) + \lambda\big(W(t)\big)\partial_x\rho(t,x) = 0, \quad t \in (0,+\infty),\ x \in (0,1), \\ \rho(0,x) = \rho_0(x), \quad x\epsilon(0,1) \\ U(t) - \rho^*\lambda(\rho^*) = k\left(\rho(t,1) + d(t)\lambda\big(W(t)\big) - \rho^*\lambda(\rho^*)\right), t\epsilon(0,+\infty) \\ \rho(t,0)\lambda\big(W(t)\big) = U(t), \quad t \in [0;+\infty) \\ W(t) = \int_0^1 \rho(t,x)dx, \quad t \in (0;+\infty) \end{cases} \quad (8)$$

where is the $\rho(t,x)$product density, $\lambda(\cdot) \in C^1([0,+\infty);\ [0,+\infty))$is the characteristic rate function, $W(t)$is the total mass, $U(t)$is the controlling and $k \in [0,1]$non-negative feedback parameter, is the $\rho^* \geq 0$equilibrium density, and is the $d(t) \in R$bounded (known) fluctuation in the scale. The weak solution of the closed loop system in Equation (8) is defined below.

Definition 1 (weak solution). *We fix that T > 0.*$\rho \in C^0([0,T];\ L^1(0,1))$ *The function (8) is called a weak solution if* $s \in (0,T]$*the following equation holds for each and each* $\varphi\epsilon C^1([0,s] \times [0,1])$*satisfying* $\varphi(s,x) = 0, \forall x \in [0,1]$*and :*$\varphi(t,1) = k\varphi(t,0), \forall t \in [0,s]$

$$\int_0^s \int_0^1 \rho(t,x)(\partial_t\varphi(t,x) + \lambda\big(W(t)\big)\partial_x\varphi(t,x)dxdt + \int_0^s ((1-k)\,\rho^*\lambda(\rho^*) + d(t))\varphi(t,0)dt + \int_0^1 \rho(0,x)\varphi(0,x)\,dx = 0$$

$d \equiv 0, \rho^* \geq 0, \mathrm{p} \in [1,+\infty)$ and $\mathrm{k} \in [0,1]$be given. Then, the existence and uniqueness of the non-negative weak solution of $\rho \in C^0([0,+\infty);$ the closed circuit system in equation (8) $L^p(0,1)$and the non-negative classical solution are in $\rho \in C^1([0,+\infty) \times [0,1])$[7,10].

Analyze the ISS for the system of equations (8) in the sense of the following definitions . $\rho^* \geq 0$It is also known as global ISS. Note that ISS Lyapunov functions can be defined in a very general way, and we refer to ([31], Definition 2.11) for such a definition. In the following definition (3), we introduce the ISS-Lyapunov functions adapted to the system of equations (8).

Definition 2 (Introduction stagnation (ISS)).$D > 0$ *let it be The equilibrium of the closed-loop system in equation (8) is* $\rho^* \geq 0$*the* L^2*-norm exponential ISS under any perturbation function* $d(\cdot) \in L^\infty(0,\infty)$*,* $\|d\|_{L^\infty(0,\infty)} \leq D$*if there dare positive constants independent* $\gamma1, \gamma2, \gamma3$*of , for each* *initial condition $\rho_0(x) \in L^2(0,1)$the closed-loop system in equation L^2- changes are introduced. (8) satisfies the following*

$\|\rho(t,\cdot) - \rho^* \|_{L^2} \leq \gamma_2 e^{-\gamma_1 t}\|\rho_0 - \rho^* \|_{L^2} + \gamma_3 \|d(s)\|_{L^\infty(0,t)}, \ t \in [0,+\infty)$.(9)

So, the $d \in D := \{d(\cdot)\epsilon L^\infty(0.\infty): \|d\|_{L^\infty} \leq D\}$ISS is in equilibrium with respect to its disturbances .ρ^*

Definition 3 (ISS-Lyapunov function). The function L : $L^2(0,1) \to R_+$is if

(i) $\alpha_1 > 0$and $\alpha_2 > 0$if there are positive constants, for all solutions, ρ $\in C^0([0,\infty))$it helps to be the ISS-Lyapunov function for the closed-loop system in equation (8), $L^2(0,1)$and t $\in [0,+\infty)$at

$$\alpha_1\|\rho(t,\cdot) - \rho^*\|_{L^2}^2 \leq \boldsymbol{L}\big(\rho(t,\cdot)\big) \leq \alpha_2\|\rho(t,\cdot) - \rho^*\|_{L^2}^2,(10)$$

(ii) η> 0and v> 0 are positive constants such that for all solutionsρ ∈ $C^0\big([0,\infty);\ L^2(0,1)\big)$; t $\in [0,+\infty)$

$$\frac{d}{dt}\boldsymbol{L}\big(\rho(t,\cdot)\big) \leq -\eta\, \boldsymbol{L}\big(\rho(t,\cdot)\big) + v d^2(t).$$

L , we, for example, ([31], Section 2.2). To simplify the definition , we also include the following function:

(11) L$(t) :=$ $\boldsymbol{L}\big(\rho(t,\cdot)\big)$,

where is ρ $\in C^0\big([0,\infty);\ L^2(0,1)\big)$ −the solution of equation (8).

Theorem 1 ($\rho^* \geq 0$for ISS). *Any$\rho^* \geq 0$, $k \in [0,1)$, $R > 0, D > 0$ and can be plotted on $\rho_0 \geq 0$(0, 1) any satisfying . $\rho_0 \in L^2(0,1)$It can also be assumed*

$$\|\rho_0(\cdot) - \rho^* \|_{L^2(0,1)} \leq R. \ (12)$$

has $\rho \in C^0\big([0,\infty);\ L^2(0,1)\big)$an almost everywhere nonnegative weak solution , λgiven by equation (6).

steady state of the system in equation (8) . is the ρ^*normally exponential ISS $d \in \{d(\cdot) \in L^\infty(0,\infty): \|d\|_{L^\infty} \leq D\}$with respect to any distortion function , where L^2 −.
The proof of this theorem is detailed in [1].

3. Numerical study of the asymptotic stability of the scalar conservation law with nonlocal velocity and measurement error

We divide the field F using an equally spaced grid with cell $J \in N$width $\Delta x J = 1[0,1]\ \Delta x$. The cell centers $x_j = \left(j - \frac{1}{2}\right)\Delta x$,are denoted by and , and $j \in \{1,\ldots,J\}$the field boundary is $x_0$and x_J, respectively. Furthermore, we $W(t)$discretize ,

$$W^n = \Delta x \sum_{j=1}^{J} \rho_j^n, \quad n \in \{1,2,\ldots\}(13)$$

Values of the solution at the $\rho_j^n = \rho(t^n, x_j)$node point . Furthermore, we λ^ndefine the discrete values by:

$\lambda^n := \lambda(W^n) = \frac{A}{B+W^n}, \ A > 0, \ B > 0,$(14)

where denotes discrete time, so the time step Δtsatisfies $t^n = n\Delta t, n \in \{0,1, \dots\}$the stationarity condition due to the Courant-Friedrichs-Lévy condition (CFL). This means that the condition $\Delta t$is selected as:

$0 < r^n := \frac{\lambda^n \Delta t}{\Delta x} \leq 1, \ \forall n \in \{0,1, \dots\}.$(15)

For all$n \geq 0$ $\lambda^n \leq \frac{A}{B}$Since , we can choose a probability small but fixed Δtsuch that the precondition (42) is true for Δtall $n$and Δx. This selection allows you to get the same mesh in time. As in the continuous case we have $\rho^* > 0$. For given initial values $\vec{\rho}^0 = (\rho_0^0, \rho_1^0, \dots, \rho_J^0)^T$, here $\rho_j^0 \geq 0, j \in \{0, \dots, J\}$we use the first-order finite volume scheme given by the revealed counterflow method to discretize the system of equations (8).

$$\begin{cases} \rho_j^{n+1} = (1 - r^n)\rho_j^n + r^n \rho_{j-1}^n, \quad j \in \{1, \dots, J\}, \ \ n \in \{0,1, \dots\}, \\ \ \ \rho_0^{n+1} = k\rho_J^{n+1} + (1-k)\frac{\rho^* \lambda(\rho^*)}{\lambda^{n+1}} + kd^{n+1}, \ \ n \in \{0,1, \dots\}. \end{cases} \quad (16)$$

4. Numerical modeling

In this section, we illustrate the theoretical results of Sections 2 and 3 by presenting numerical calculations of the ISS scalar conservation law with nonlocal velocity and bounded measurement error. We apply the discretization introduced in the previous section and $A = B = 1$, resulting in a characteristic speed function.

$\lambda\big(W(t)\big) = \frac{1}{1+W(t)}$, in this $W(t) = \int_0^1 \rho(t,x)dx$(17)

As a measurement error, we consider the following:

$d(t) = 2.4 \times 10^{-3} \sin(t), \ t \in (0, \infty)$(18)

Example 1

In this example $x \in [0,1]$, the $\rho^* = 0$equilibrium solution for and $\rho_0(x) = 1 + \sin(2\pi x)$we consider the initial condition. In the figures below, the CFL condition is equal to 0.5 (respectively 0.9 in Table 1) and the system of equations (8) for the two given conditionsL^2 $||\vec{\rho}^n - \rho^*||_{\ell^2}$l will show the decay of the discrete error in Here is $= a \leq 1$a stronger condition than CFL (15) and it Δtmeans that:

$\lambda_W^n \frac{\Delta t}{\Delta x} \leq a \leq 1, \ n \geq 0.$ (19)

$CFL \leq 1$ value improves the stability of the circuit due to additional artificial dispersion of the circuit. Due to artificial diffusion and fluctuation, we Δxobserve only approximately excluded first-order convergence with respect to the wind-driven scheme. Figure 1 shows the convergence of the weak solution of equation (8) $k$to equilibrium for different values of . As expected, $k$we observe that the rate of decay of the Lyapunov function decreases as it increases. We also $\Delta x = 10^{-3}$see that no additional error is observed other than below the mesh resolution.

Table 1. $||\vec{\rho}^{n} - \rho^{*}||^{2}_{\ell^2}$ comparing with the values at different J-node points of the grid $\rho^{*} = 0$, k = 0.3 and T = 10

(a) CFL = 0.5.		
J	$\|\vec{\rho}^{n} - \rho^{*}\|_{\ell^2}$	order
100	1.9171 e-05	–
200	1.1899 e-05	0.6881 e+00
400	6.9631 e-06	0.7730 e+00
800	3.7638 e-06	0.8875 e+00
1600	1.5902 e-06	1.2430 e+00
(b) CFL = 0.9.		
J	$\|\vec{\rho}^{n} - \rho^{*}\|_{\ell^2}$	order
100	1.3831 e-05	–
200	8.1304 e-06	0.7665 e+00
400	4.8604 e-06	0.7423 e+00
800	2.8262 e-06	0.7822 e+00
1600	1.1624 e-06	1.2818 e+00

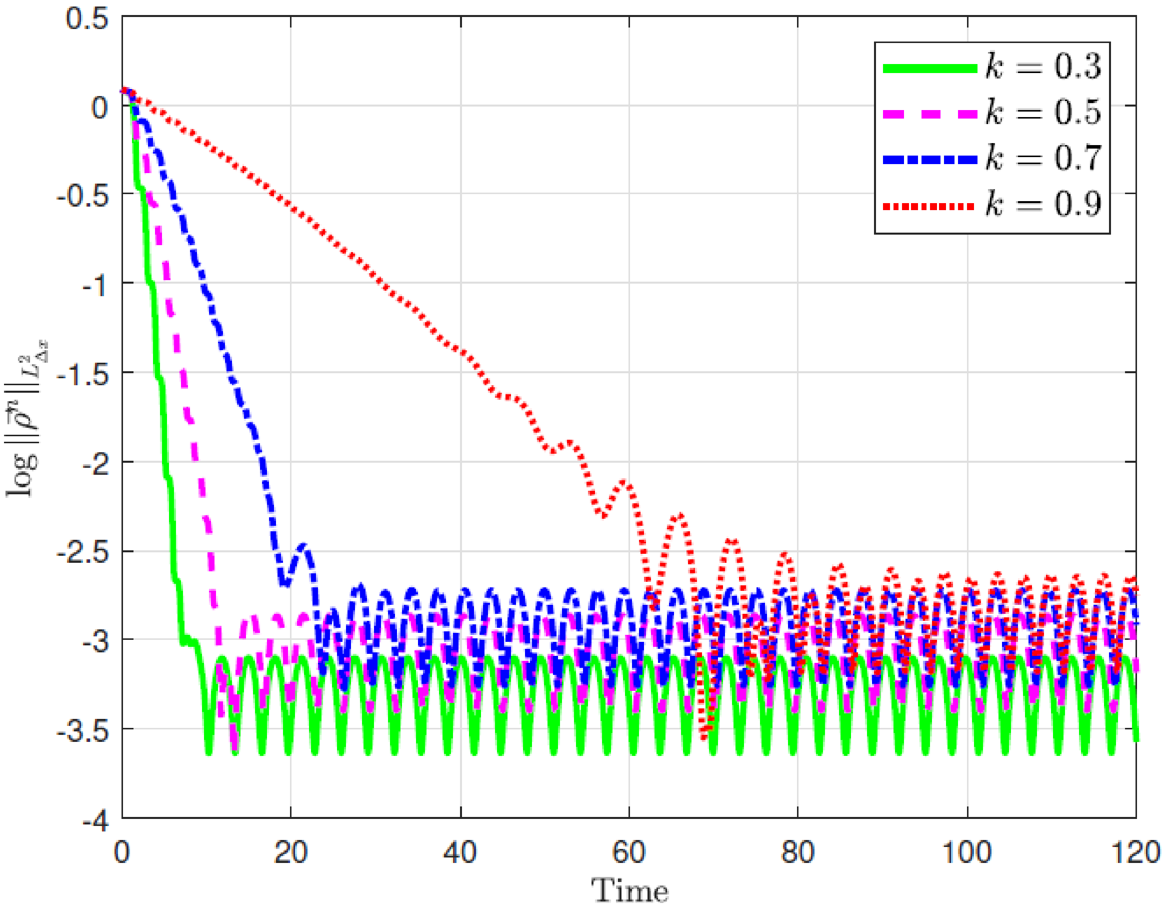

Figure 1.$||\vec{\rho}^n - \rho^*\vec{e}||_{L^2_{\Delta x}}$ comparing the logarithmic scale with Courant-Friedrichs-Levy (CFL)= 0.75 and $\rho^* = 0$ condition.

SUMMARY AND FUTURE PLANS

The input data consistency condition (ISS) for a scalar conservation law with nonlocal velocity and bounded measurement error. The ISS-Lyapunov function is used to test the ISS stability conditions for a scalar conservation law with nonlocal velocity and measurement error. The numerical solution of the error decay of the ISS-Lyapunov function was analyzed. Finally, numerical simulations show theoretical results.

An extension that can also be considered is the ISS with respect to the L^2 norm in time in the continuous and discrete case .

The drawback of Theorem 1 is that the system may not have an a priori solution. As noted in Note 1, for the presented problem, time and It may be possible to extend the results of [13-15] to obtain a continuous solution on the $L^{2\text{-}norm.}$ This is the subject of future work.

REFERENCES

1. Armbruster, D.; Degond, P.; Ringhofer, C. A model for the dynamics of large queuing networks and supply chains. SIAM J. Appl.Math. 2006 , 66, 896–920, doi :10.1137 /040604625.
2. Herty, M.; Klar, A.; Piccoli, B. Existence of solutions for supply chain models based on partial differential equations. SIAM J.
3. Math. Anal. 2007 , 39, 160–173, doi :10.1137 /060659478.
4. He, F.; Armbruster, D.; Herty, M.; Dong, M. Feedback control for priority rules in re-entrant semiconductor manufacturing. Appl.Math. Model. 2015 , 39, 4655–4664, doi:10.1016/j.apm.2015.03.061.
5. D'Apice, C.; Göttlich, S.; Herty, M.; Piccoli, B. Modeling, Simulation, and Optimization of Supply Chains; Society for Industrial and Applied Mathematics (SIAM): Philadelphia, PA, USA, 2010; pp. x+206. A continuous approach, doi:10.1137/1.9780898717600.
6. Chen, H.; Harrison, JM; Mandelbaum, A.; Van Ackere, A.; Wein, LM Empirical Evaluation of a Queuing Network Model for
7. Semiconductor Wafer Fabrication. Opera. Res. 1988 , 36, 202–215, doi:10.1287/opre.36.2.202.
8. Helbing, D. Traffic and related self-driven many-particle systems. Rev. Modern Phys. 2001 , 73, 1067–1141, doi:10.1103/RevModPhys.73.1067.
9. Coron, JM; Wang, Z. Output Feedback Stabilization for a Scalar Conservation Law with a Nonlocal Velocity. SIAM J. Math. Anal. 2013 , 45, 2646–2665, doi :10.1137 /120902203.
10. Coron, JM; Kawski, M .; Wang , Z. Analysis of a conservation law modeling a highly re-entrant manufacturing system. Discreet. Contin. Dyn. Syst. Sir. B 2010 , 14, 1337–1359, doi:10.3934/dcdsb.2010.14.1337.
11. Shang, P.; Wang, Z. Analysis and control of a scalar conservation law modeling a highly re-entrant manufacturing system. J. Differ. Equ. 2011 , 250, 949–982, doi:10.1016/j.jde.2010.09.003.
12. Chen, W.; Liu, C.; Wang, Z. Global Feedback Stabilization for a Class of Nonlocal Transport Equations: The Continuous and Discrete Case. SIAM J.

Control Optim. 2017 , 55, 760–784, doi :10.1137 /15m1048914.
13. Tanwani, A.; Prieur, C.; Tarbouriech, S. Stabilization of linear hyperbolic systems of balance laws with measurement errors. In Control Subject to Computational and Communication Constraints; Springer: Cham, Switzerland, 2018; Volume 475, pp. 357–374. Lect. Notes Control Inf. Sci.
14. Dos Santos, V.; Bastin, G.; Coron, JM; d'Andréa Novel, B. Boundary control with integral action for hyperbolic systems of conservation laws: stability and experiments. Automatic. J. IFAC 2008 , 44, 1310–1318, doi:10.1016/j.automatica.2007.09.022.
15. Bastin, G.; Coron, JM; d'Andréa Novel, B. On Lyapunov stability of linearized Saint-Venant equations for a sloping channel. Netw. Heterog. Media 2009 , 4, 177–187, doi:10.3934/nhm.2009.4.177.
16. Bastin, G .; Coron, JM Exponential stability of semi-linear one-dimensional balance laws. In Feedback Stabilization of Controlled Dynamical Systems; Springer: Cham, Switzerland, 2017; Volume 473, pp. 265–278. Lect. Notes Control Inf. Sci.
17. Bastin, G.; Coron, JM Stability and Boundary Stabilization of 1-D Hyperbolic Systems; Springer: Berlin/Heidelberg, Germany, 2016, doi :10.1007 /978-3-319-32062-5.
18. Coron, JM; d'Andréa Novel, B.; Bastin, G. A strict Lyapunov function for boundary control of hyperbolic systems of conservation laws. IEEE Trans. Automatic. Control 2007 , 52, 2–11, doi:10.1109/TAC.2006.887903.
19. Diagne, A.; Bastin, G.; Coron, JM Lyapunov exponential stability of 1-D linear hyperbolic systems of balance laws. Auto. J. IFAC 2012 , 48, 109–114, doi:10.1016/j.automatica.2011.09.030.
20. Gugat, M.; Perrollaz, V.; Rosier, L. Boundary stabilization of quasilinear hyperbolic systems of balance laws: Exponential decay for small source terms. J. Evol. Equ. 2018 , 18, 1471–1500, doi :10.1007 /s00028-018-0449-z.
21. Cen, LH; Xi, YG Stability of boundary feedback control based on weighted Lyapunov function in networks of open channels. Acta Automat. Sinica 2009 , 35, 97–102. doi:10.3724/SP.J.1004.2009.00097.
22. Prieur, C.; Mazenc, F. ISS-Lyapunov functions for time-varying hyperbolic systems of balance laws. Math. Control Signal Syst. 2012 , 24, 111–134, doi:10.1007 /s00498-012-0074-2.