Software Complexes, Numerical Techniques, and Mathematical Modeling

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https://doi.org/10.5281/zenodo.10471397

Keywords: optimization, rescheduling, modeling, intensity, martingale, and just-in-time.
Abstract: This article examines the models of basic Just-In-Time (JIT) systems using point processes in reverse time. This method permits certain presumptions regarding the workings of actual systems. We thus formulate and solve a few very basic optimal control problems for a system with bounded intensity and for a multi-stage just-in-time system. For the objective functions, results are computed as expected linear or quadratic forms of the trajectories' deviations from the intended values. The statements' proofs employ the martingale method. In logistics tasks, just-in-time systems are frequently taken into consideration, and only (or mostly) deterministic methods are used to describe them. Nonetheless, it is evident that stochastic events are frequently observed in these systems and the related processes.
It is crucial to identify strategies for the best just-in-time process management in these kinds of stochastic situations. In this paper, we propose to use martingale methods for this description. Here, straightforward methods for stochastic JIT process optimization are shown.

I. Introduction.

In this article, we consider some stochastic models of simple just-in-time systems. The well-known principle of just-in-time system (abbreviated as JIT system) is used in many areas. Examples include just-in-time production systems, pedagogical strategies of just-in-time teaching, and just-in-time compilation methods in computer programming. It should be noted that at present mathematical, especially stochastic, models for JIT systems are not sufficiently developed. Such models are necessary for solving optimal control problems, which could allow optimizing the allocation of system resources and implementing optimal planning of a stochastic JIT system. The purpose of this article is to present an approach to the stochastic description of JIT systems, which would be suitable for both analytical methods and computer simulation. Mathematical models of such systems should allow assuming that the trajectories of processes must take the given values at a fixed time. Such behavior of processes is known in stochastic bridges and stochastic processes in the reverse time. Thus, we should consider models of systems with the requirement of JIT in terms of processes with the behavior of trajectories close to stochastic bridges. Models should also allow investigating possible violations of this requirement that are unavoidable for real systems. Stochastic process time reversals have been the subject of research for many years. We observe that the study of these processes is the focus of several works pertaining to stochastic bridges. Furthermore, some studies on reversible Markov processes also adjoin process descriptions in the opposite direction. In this paper, we examine semi martingale models of elementary JIT systems for point processes near the previously mentioned Poisson bridge. Here, we'll grant some suppositions regarding the workings of actual systems. Thus, a system with bounded intensity and simple cases of multi-stage JIT systems are examined. As demonstrated, it is possible to formulate and solve basic optimal control problems for these situations. The semi martingale technique is used in the results proofs.

II. Materials and Methods

A basic JIT system's time reversal technique. Think about a Just-In-Time (JIT) system that can be explained using point processes, such as counting. We assume that, starting from the zero moment, a fixed time $T > 0$ must be met by an integer number K of operations within the system. This indicates that for every time t ∈ [0, T], the number of operations left, tr, is equal to the number p minus the value p_t of a counting process, t = (p_t)_t>0: tr = p_t − p_t.
We now give a formal description of the mathematical model. Let (Ω, F, P) be a probability space populated with a nondecreasing right-continuous family of σ-algebras F = (Ft)t>0, complete with respect to P (i.e., the conditions of [13] hold). On the stochastic basis B = (Ω, F, F = (Ft)t>0, P) the process X = (Xt)t>0 is supposed to be the point process with trajectories in the Skorokhod space, Xt ∈ N0 = {0, 1, 2, . . . } and $\Delta Xt = Xt - Xt-$ ∈ {−1, 0}
The process X can be represented as a difference: X = $X0 - N = K - N$, where $N = (Nt)t$>0 is the counting process of the number of negative jumps of X, with the initial value $X0$ = K > 0 (i.e., K ∈ N = {1, 2, . . . }, $N0$ = 0, and $Xt = K - Nt$, for all t > 0).
We suppose that the submartingale N and supermartingale X on B admit the well-known Doob–Meyer decompositions (see, e.g., [13]):
$Nt = N\tilde{}\ t + mN\ t$, $Xt = X\tilde{}\ t - mN\ t$ (1) with the compensators $N\tilde{}$ = ($N\tilde{}\ t$) t>0 and $X\tilde{}$ = ($X\tilde{}\ t$)t>0,
and the square-integrable martingale mN = ($mN\ t$) t>0 with the quadratic characteristic $\langle mN \rangle\ t = N\tilde{}\ t$ for all t > 0.

We also suppose in this article that $\tilde{N}_t = \int_0^t (K - N_s) \cdot \frac{1}{T - s} \cdot \mathrm{I}\{s < t\}\, ds$, (2) where $\mathrm{I}\{\cdot\}$ is an indicator function (i.e., I{true} = 1, I{false} = 0). From (1) and (2) it follows that the process X has the decomposition: $X_t = K - \int_0^t X_s \cdot \frac{1}{T - s} \cdot \mathrm{I}\{s < t\}\, ds - m^N_t$. (3)

In the general case, for the basic model we assume that the point process X admits the representation: $X_t = K - \int_0^t h_s\, ds + m^X_t$.

With the intensity of negative jumps $h = h(X) = (h_t(X))_{t>0}$ and the martingale $m^X = (m^X_t)_{t>0}$. In the particular case (3), the following equality holds:

$h_t = h_t(X) = X_t \cdot \mathrm{I}\{s < t\}/(T - s)$, (5) and $m^X = -m^N$, i.e. $m^X_t = -m^N_t$ for all $t > 0$.

It is well known that the compensator of the point process defined by formula (2) corresponds to the bridge of a Poisson process.

Consider a standard Poisson process $\pi = (\pi_s)_{s \in [0,T]}$ on the stochastic basis B with the initial value $\pi_0 = 0$ and any positive intensity $\lambda > 0$. Let $\mathrm{F}^0_t = \sigma\{\pi_s : T - t \leqslant s \leqslant T\}$ for $t \in [0, T]$, $\mathrm{F}^0_t = \mathrm{F}^0_T$ for $t > T$, and nondecreasing family σ-algebras $\mathrm{F} = (\mathrm{F}_t)_{t>0}$ be the right continuous completion of $(\mathrm{F}^0_t)_{t>0}$.

Define the reverse time supermartingale $Y = (Y_t)_{t>0}$ as $Y = \pi_{T-t}$ for $t \in [0, T]$ and $Y_t = \pi_0 = 0$ for $t > T$. Then Y is F-adapted and it has the decomposition (as it easily follows, from Theorem 2.6 in [8]): $Y_t = \pi_T - \int_0^t \frac{Y_s}{T - s} \cdot \mathrm{I}\{s < t\}\, ds + m^Y_t$, (6) where $m^Y = (m^Y_t)_{t>0}$ is a square-integrable martingale with the quadratic characteristic $\langle m^Y \rangle_t = \int_0^t \frac{Y_s}{T - s} \cdot \mathrm{I}\{s < t\}\, ds$.

The comparison of (3) and (6) illustrates the fact known for bridge processes: the representation of the process $X = K - N$ (with the initial value K and the Poisson bridge N) coincides with the reverse time representation Y of the Poisson process π (with any strictly positive intensity λ) under the condition for the initial value $Y_0 = \pi_T = K$. Thus, we can consider the behavior of the trajectories of the process X with $X_0 = K$ and $X_t = 0$ for $t > T$ as the embodiment of the just-in-time requirement.

Therefore, the main idea of the presented description of JIT systems is the realization of the corresponding behavior of trajectories by means of proper control of $h = (h_t)_{t>0}$, which is the intensity of the negative jumps of X in the base model (4). This intensity can be regarded as a negative feedback tending to $-\infty$ as $t \to T$ in the case of nonzero X_t.

Note that in (6) it does not directly depend on the intensity λ of the initial process π. The distribution of the main process X in (4) is determined by the intensity of the negative jumps h, which in the particular case of (5) depends on the values of K and $T > 0$. Along with X, we define for the base model (4) the auxiliary functions for $\mathrm{E}X_t$, $\mathrm{E}X^2_t$ and $\mathrm{E}(X_t - R_t)^2 = G_t - R^2_t$ (i.e., for the mean, the second moment, and the variance of X, respectively).

For the functional $h = h(X)$ of general form in (4), and the initial value K, it is assumed that $R_t = R_t(K; h) = \mathrm{E}X_t$, $G_t = G_t(K; h) = \mathrm{E}X^2_t$, $V_t = V_t(K; h) = \mathrm{E}(X_t - R_t)^2$. (7)

In the particular case (5), these functions depend only on the values of t, K, and T. Therefore, for (5) we use the notations: $r_t(K; T) = R_t = \mathrm{E}(X_t / X_0 = K; X_t = 0)$, (8) $g_t(K; T) = G_t = \mathrm{E}(X^2_t / X_0 = K; X_t = 0)$, (9) $v_t(K; T) = V_t = \mathrm{E}((X_t - R_t)^2 | X_0 = K; X_t = 0) = g_t(K; T) - r_t(K; T)^2$. (10)

For the functions (8), (9) and (10) defined for X in (4) with the intensity (5), we have $r_t(K; T) = K \cdot \frac{T - t}{T} \cdot \mathrm{I}\{t \leqslant T\}$,

$g_t(K; T) = \left(K \cdot \frac{T - t}{T} \right)^2 \cdot \mathrm{I}\{t \leqslant T\} + K \cdot \frac{(T - t) \cdot t}{T^2} \cdot \mathrm{I}\{t \leqslant T\}$, (12) $v_t(K; T) = K \cdot \frac{(T - t) \cdot t}{T^2} \cdot \mathrm{I}\{t \leqslant T\}$.

Problems of optimal planning for a multi-stage JIT process. Consider a model of simple multi-stage JIT systems in terms of the proposed description. We assume that it is a set of separate processes in reverse time (or bridges of corresponding processes) with a single aggregate plan. This section presents a simple solution to the problem of the optimal times for changing the stages for the model. In the cases considered here, the mean-square deviations of the trajectories from the planned values are minimized. In addition, we consider the problem of optimal rescheduling for the case of two stages and for its multistage generalization. 2.1. Separate processes in reverse time. Let us consider optimal control problem for the following scheduling model. Let the execution of $(K + 1)$ operations in time T be subdivided into $n \in \mathbb{N}$ stages: every successive $K(i)$ operations must be performed in stage i, which lasts the time $\varsigma(i)$, for all $i = 1, 2, \ldots, n$.

The following conditions for the time and number of operations must be fulfilled:

$\sum_{i=1}^n \varsigma(i) = T$, (14) $\sum_{i=1}^n K(i) = K$. (15)

We also define the condition for the uniformity of the operations:

$K(i) = K(i) \cdot \varsigma(i)/T$ for all $i = 1, 2, \ldots, n$.

Thus, the model of this JIT system is a set of separate processes in reverse time (or of proper bridges). Suppose that we must insure the uniform fulfillment of the plan $\varsigma = \{\varsigma(1), \varsigma(2), \ldots, \varsigma(n)\}$ in the sense of , minimizing the weighted variance of the deviation from it.

We consider the problem of finding an optimal plan $\varsigma^* = \{\varsigma^*(1), \varsigma^*(2), \ldots, \varsigma^*(n)\}$ for which $\Phi(\varsigma^*) = \inf_\varsigma \Phi(\varsigma)$, (17) where the objective function $\Phi(\varsigma)$ is the sum of weighted variances (10) for the processes in (4) with initial values $K(i)$ and times of performance $\varsigma(i)$, $i = 1, 2, \ldots, n$: $\Phi(\varsigma) = \sum_{i=1}^n \alpha(i) \cdot \int_0^{\varsigma(i)} v_t(K(i), \varsigma(i))\, ds$ (18) under conditions (14) and (15), and for strictly positive weights: $\alpha(i) > 0$ for all $i = 1, 2, \ldots, n$.

Theorem 1. For the plan that minimizes the objective function $\Phi(\varsigma)$, $\varsigma^*(i) = T \cdot \left\{ \alpha(i) \cdot n \cdot \sum_{j=1}^n 1/\alpha(j) \right\}^{-1/2}$ for all $i = 1, 2, \ldots, n$. (20) Remark 1. Theorem 1 implies the trivial consequence that for equal weights the equal times are optimal: for $\alpha(1) = \alpha(2) = \ldots = \alpha(n) > 0$, $\varsigma^*(i) = T/n$ for all $i = 1, 2, \ldots, n$. (21) 2.2. The problem of optimal rescheduling for a two-stage JIT process. As it follows from (21), for $n = 2$, in the case of equal weights, it then holds that $\varsigma^*(1) = \varsigma^*(2) = T/2$.

But in real systems, rescheduling—a process for reviewing the plan while it's being implemented—occurs in addition to a priori stage planning. The JIT system's operations in this instance are carried out in line with the process intensity in (3) for the planned initial value **N** and the planned time T for the following cases: **h** $\in$ [0, σ], $\sigma \in [0, T]$, where σ is the rescheduling time. Thus, the initial plan with the values of K and T is executed in the first stage for $t \in [0, \sigma]$. The subsequent re-planning process is put into action at time τ. The time interval $[\sigma, T]$ is when the second stage is completed. Here, following the rescheduling, the new execution time $(T - \sigma)$ and the starting value of the number of operations $(X\sigma)$ are set in the interval $[\sigma, T]$ for the new process in reserve time.
plan in the first stage and the deviation from the new plan in the second stage. Thus, we consider the problem of finding an optimal value σ * for which $\Psi(\sigma$ *) = inf σ $\Psi(\sigma)$, (23) where the objective function $\Psi(\sigma)$ is the integrated variance (7) for the intensity $h = h(X)$ is equal to $\Psi(\sigma) = \int T\ 0\ Vt(K; h)\ ds$. (24) Here the intensity for the rescheduling is equal to $ht(X) = h$ (1) t $(X) \cdot$ I$\{t \in [0, \sigma)\} + h$ (2) t $(X) \cdot$ I$\{t \in [\sigma, T)\}$, (25) where h (1) t $(X) = Xt/(T - t)$, $h(2)$ t $(X) = Xt/(T - \sigma - t)$. (26) Lemma 2. For the time σ that minimizes the objective function $\Psi(\sigma)$, σ * = T /3.

The problem of the optimal level of resources of a simple system with possible violations of the condition. In this section, we consider some assumptions about violations of the JIT condition in processes inherent in real systems. Thus, we assume that the intensities of point processes can be bounded. We note that such a representation of the process X in (4) does not correspond to the time reversal procedure for a point process with fixed initial value. Nevertheless, such a representation in terms of point processes is useful for describing a controlled system with a violation of the condition of JIT. For such a model, the task of optimal control arises – to find the value of the maximum level of intensity of the point process for each operation under conditions of payment for the value of this boundary, and payment for non-compliance with the JIT requirement. We suppose that the intensity h in (4) can be represented as $ht = ht(X) = Xt \cdot \min\{\Lambda, \mathrm{I}\{t < T\}/(T - t)\}$, (32) where $\Lambda \in [0, \infty)$ is a finite maximum level of intensity for each operation. Under this assumption for h, the JIT-condition $XT = 0$ may not hold, and obviously

$$\mathrm{P}\{\omega : XT\ (\omega) > 1\} > 0 \text{ and } \mathrm{E}XT > 0.$$

We assume that the payment for this violation of the JIT condition is proportional to the mean value of the number of uncompleted operation EXT. The coefficient of proportionality is denoted by α. The greater the upper level Λ, the smaller the value of EXT and the closer to the fulfillment of the JIT requirement. Since the resources of the real system provide the level Λ, it also has a certain positive cost with a proportionality factor of β. Moreover, Λ can serve as a control parameter in the system (4). Thus, we consider the problem of optimal control of the process X in (4) for fixed $K \in$ N and $T > 0$, and under the assumption for h. It is necessary to find an optimal value Λ * for which the problem is analogous to the problems (17), (23) and (29): $\Theta(\Lambda^*)$ = inf $\Lambda > 0$ $\Theta(\Lambda)$, (33) where the objective function $\Theta(\Lambda)$ is equal to $\Theta(\Lambda) = \alpha \cdot \mathrm{E}XT + \beta \cdot \Lambda$ (34) under the conditions: $\alpha > 0$, $\beta > 0$. (35) Theorem
For the maximum intensity level, which minimizes the objective function

$\Theta(\Lambda)$, Λ * = $\sqrt{\ }\ \alpha \cdot K\ \beta \cdot e \cdot T$ if $\alpha \cdot K \cdot T/\beta \in [e, +\infty)$, (36) Λ * = $[\log(\alpha \cdot K \cdot T/\beta)]/T$ if $\alpha \cdot K \cdot T/\beta \in (1, e)$, (37) and Λ * = 0 if $\alpha \cdot K \cdot T/\beta \in (0, 1]$.

III. Discussion.

The main purpose of this article is to show the possibilities of using of the time reversal approach in problems concerning just-in-time. We demonstrate simple methods for optimizing JIT systems, for the case of a point (counting) process, represented in semimartingale terms. We also note that the statements of Theorem 1, Lemma 2, and Theorem 2 are valid in the case of a random walk in reverse time (Lemma 1 and Theorem 2 remain true if the coefficients are properly replaced). In this case, the semimartingale representation methods and optimal control problems are close. In the case of nonstationary processes in direct time, the results are also anticipated. Finally, note that the method of representing JIT systems discussed in the article in terms of predictable semimartingale characteristics creates opportunities for simple and clear computer modeling. Obviously, the simulation is easy to implement on the basis of the infinitesimal relation for

$$X: \mathrm{P}\{\Delta Xt = Xt{+}\Delta - Xt = -1|\mathrm{F}t\} = ht(X) \cdot \Delta + o(\Delta) \text{ as } \Delta \to 0, \text{ for all } t > 0.$$

Thus, it follows that the discussed approach can serve as an initial step for the analysis of stochastic JIT systems.

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